

# Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent acceleration

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## ABSTRACT

A new phenomenon of coherent acceleration of ions by a discrete spectrum of electrostatic waves propagating perpendicularly to a uniform magnetic field is described. It allows the energization of ions whose initial energies correspond to a region of phase space which is below the chaotic domain. The ion orbits below the chaotic domain are described very accurately using a perturbation analysis to second order in the wave amplitudes. This analysis shows that the coherent acceleration takes place only when the wave spectrum contains at least two waves whose frequencies are separated by an amount close to an integer multiple of the cyclotron frequency. The way the ion energization depends on the wavenumbers and wave amplitudes is also presented in detail using the results of the perturbation analysis.

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## I. INTRODUCTION

In this paper we study the dynamics of ions interacting with a discrete spectrum of electrostatic waves propagating perpendicularly to a uniform magnetic field. As shown in Refs. 1 and 2, such a study is also applicable when the direction of propagation of the waves departs by a small angle from perpendicular propagation. In the case when the ions interact with a single electrostatic wave it has been shown in Refs. 1-4 that the ions gain energy, on the average, only if their motion is chaotic. This occurs for electric field amplitudes above a threshold value.<sup>1-4</sup> In a weakly nonuniform magnetic field the threshold for chaotic dynamics may even be reduced.<sup>5</sup> The energization of ions, in the chaotic phase space of a single wave, was found to be applicable in explaining the generation of energetic ion tails in lower-hybrid heating experiments in tokamaks.<sup>6,7</sup> Experiments have also demonstrated the chaotic energization of plasma ions by single intense electrostatic waves and, furthermore, verified some of the details of the theoretical results for ion dynamics in a single wave.<sup>8</sup> The threshold for chaotic ion heating has been observed, and it has been determined that electrostatic ion Bernstein waves are responsible for rapid changes in the ion distribution function for wave amplitudes above threshold.<sup>8</sup> In this paper we present a general theory of the dynamics of ions in a discrete, multi-wave spectrum propagating across a uniform magnetic field. The results of this study can serve as a basis for new means of energizing, or extracting energy from, ions in a plasma. A forthcoming paper<sup>9</sup> will be dedicated to the application of these results to the particular case of transverse energization of ions in the ionosphere observed to occur in lower hybrid solitary structures,<sup>10</sup> but here we will not discuss any of the possible

applications.

The main topic of this paper is to determine the energy an ion can gain from multiple waves, and the nature of the dynamics of the ion while it is being energized. In the case of one wave whose frequency is above the ion-cyclotron frequency, the ion acceleration occurs only in the bounded region of chaotic phase space. This entails a lower bound in the wave amplitude, as well as in the initial ion energy,<sup>1,11</sup> for the ion to be accelerated; it also implies that the maximum energy an ion can achieve corresponds to the upper limit of the chaotic domain in phase space.

In the case of more than one wave above the ion-cyclotron frequency, we show that, depending on the parameters of the wave spectrum, there can be coherent acceleration. Consequently, an ion may be accelerated regardless of its initial energy, and, in particular, it may be accelerated even if its initial energy corresponds to a region of phase space which is below the chaotic domain of phase space. This coherent acceleration of an ion below the chaotic region is described in detail, using a perturbation analysis carried out to second order in the waves amplitudes. The dependence of the ion energization on the wave spectrum characteristics is also discussed in detail.

The paper is organized as follows, in section II we derive the equations of motion and the Hamiltonian of the dynamics in action-angle variables of the zero-electric field case. In section III we re-state the known results in the one-wave case. Section IV describes the coherent acceleration of low-energy ions. In particular we show how the results obtained in the case of two waves easily generalize to an arbitrary discrete wave spectrum. The last section summarizes the results.

## II. EQUATIONS OF MOTION AND HAMILTONIAN FORMULATION

The motion of an ion of mass  $m$  and charge  $q$  in a uniform magnetic field  $\vec{B} = B_0 \hat{z}$ , and being perturbed by a spectrum of electrostatic waves  $\vec{E} = \hat{x} \sum_{i=1}^N E_i \sin(k_i x - \omega_i t + \varphi_i)$ , is given by

$$\frac{d^2 x}{dt^2} + \Omega^2 x = \frac{q}{m} \sum_{i=1}^N E_i \sin(k_i x - \omega_i t + \varphi_i) \quad (1)$$

where  $\Omega = qB_0/m$  is the cyclotron angular frequency. We normalize time to  $\Omega^{-1}$  and length to  $k_1^{-1}$ , and define the dimensionless variables  $X = k_1 x$ ,  $\tau = \Omega t$ . We then switch to the normalized action-angle variables of the linear oscillator. The action is  $I = X^2/2 + \dot{X}^2/2$ , where  $\dot{X} = dX/d\tau$ . The angle  $\theta$  is defined by  $X = \rho \sin \theta$ ,  $\dot{X} = \rho \cos \theta$ , where  $\rho = \sqrt{2I}$  is the normalized Larmor radius. In action-angle variables  $(I, \theta)$ , the Hamiltonian corresponding to (1) is

$$H = I + \sum_{i=1}^N \frac{\varepsilon_i}{\kappa_i} \cos(\kappa_i \rho \sin \theta - \nu_i \tau + \varphi_i) \quad (2)$$

where  $\kappa_i = k_i/k_1$ ,  $\nu_i = \omega_i/\Omega$ , and  $\varepsilon_i = (k_1 q E_i)/(m \Omega^2)$ . The Hamiltonian (2) will be the starting point of all the analytical calculations made to describe the dynamics defined by (1).

## III. RESULTS OF ONE WAVE CASE

It has been shown in Refs. 1–4, and 12, that an ion can gain energy from one wave whose frequency is above the cyclotron frequency only if the ion dynamics are chaotic. In the single



wave-particle interaction, the dynamics is qualitatively different depending on whether the wave frequency is an integer multiple of the ion cyclotron frequency (on-resonance case) or not (off-resonance case).

In the off-resonance case, it has been shown in Refs. 1 and 2 that chaotic dynamics occurs if the wave amplitude exceeds a threshold amplitude:

$$\varepsilon > \varepsilon_{th} \approx \nu^{2/3}/4 \quad (3)$$

For amplitudes above the threshold value, the chaotic phase space is bounded from below and from above in action; the lower bound of the stochastic region is

$$\rho \approx \nu - \sqrt{\varepsilon} \quad (4)$$

Since the acceleration of ions by a single wave can only be stochastic, (4) yields the minimum energy an ion needs to have in order to be accelerated by the off-resonance wave.

In the case of one on-resonance wave<sup>4,12</sup>, there is a web-structure in phase-space that extends up to infinite values of the energy. However, the web becomes increasingly thin at large energies. Also, the web structure has a lower bound in energy.<sup>11</sup> This implies that, in the on-resonance case, the initial energy of an ion needs to be high enough for the ion to be accelerated by the wave. It has been shown that this lower bound tends to lift up to higher energies as the wave amplitude is increased until crossing over with (4).<sup>11</sup> Nevertheless, for high harmonics of the cyclotron frequency, the estimate given in (4) for the lower bound in energy is approximately valid also in the on-resonance case. So, for high ion-cyclotron

harmonics, the estimate (4) can be considered to apply to the on-resonance case.

#### IV. ACCELERATION OF LOW-ENERGY IONS WITH MORE THAN ONE WAVE

In the case of more than one wave, the lower bound (4) for stochastic acceleration remains valid, if  $\nu$  is the minimum of the  $\nu_i$ 's. However, unlike for the case of one wave, the energy of an ion can vary a lot even if its initial Larmor radius is less than the limit (4) (see Fig. 1). Hence, an ion below the stochastic domain can be accelerated. The motion of an ion in this domain of phase space is regular, as can be seen on Fig. 1, and can thus be deduced from an integrable Hamiltonian. This integrable Hamiltonian can be derived from the complete Hamiltonian (2) using perturbation theory. We perform the perturbation analysis only up to second order, using the formalism of the Lie transform. This leads to an integrable Hamiltonian  $\tilde{H}$  whose orbits are found by solving  $\tilde{H} = \text{const.}$  As can be seen in Fig. 1, the orbits of  $\tilde{H}$  provide a very accurate description of the actual motion. Hence, the dynamics defined by  $H$  will be described by studying the integrable dynamics given by  $\tilde{H}$ , whose derivation is detailed in the following subsections.

##### A. Coherent acceleration in the case of two off-resonance waves

In this subsection we focus on the dynamics defined by (2), in the case of two waves such that neither  $\nu_1$  nor  $\nu_2$  are integers, and for values of  $\rho$  less than  $\min(\nu_1, \nu_2)$ . As already

mentioned before, these dynamics can be accurately described by a perturbation analysis on the Hamiltonian (2) up to second order in the wave amplitudes. The perturbation analysis is performed using the formalism of the Lie transform<sup>13,14</sup> (detailed calculations using the Lie transform in cases similar to ours can be found in Refs. 11 and 15). It defines a new set of conjugate coordinates,  $(\tilde{I}, \tilde{\theta})$ , in which the Hamiltonian (2) is transformed to

$$\begin{aligned}
\tilde{H}_2^{(off)} = & \tilde{I} + \varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho}) + \varepsilon_1^2 \delta_1 \cos[2\nu_1(\tilde{\theta} - \tau) + 2\varphi_1] S_3(\tilde{\rho}) \\
& + \varepsilon_2^2 \delta_2 \cos[2\nu_2(\tilde{\theta} - \tau) + 2\varphi_2] S_4(\tilde{\rho}) \\
& + \varepsilon_1 \varepsilon_2 \delta_3 \cos[(\nu_1 + \nu_2)(\tilde{\theta} - \tau) + \varphi_1 + \varphi_2] S_5(\tilde{\rho}) \\
& + \varepsilon_1 \varepsilon_2 \delta_4 \cos[(\nu_1 - \nu_2)(\tilde{\theta} - \tau) + \varphi_1 - \varphi_2] S_6(\tilde{\rho})
\end{aligned} \tag{5}$$

where  $\delta_1, \delta_2, \delta_3$ , and  $\delta_4$ , are unity if  $2\nu_1, 2\nu_2, (\nu_1 + \nu_2)$ , and  $(\nu_1 - \nu_2)$  are integers, respectively, and 0 otherwise, and

$$S_2(\tilde{\rho}) = \frac{1}{2\kappa\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa\tilde{\rho}) J'_m(\kappa\tilde{\rho})}{\nu_2 - m} \tag{6}$$

$$S_4(\tilde{\rho}) = \frac{1}{2\kappa\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa\tilde{\rho}) J'_{2\nu_2-m}(\kappa\tilde{\rho})}{\nu_2 - m} \tag{7}$$

$$S_5(\tilde{\rho}) = \frac{1}{2\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{m J_m(\tilde{\rho}) J'_{\nu_1+\nu_2-m}(\kappa\tilde{\rho})}{\nu_1 - m} + \frac{1}{2\kappa\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa\tilde{\rho}) J'_{\nu_1+\nu_2-m}(\tilde{\rho})}{\nu_2 - m} \tag{8}$$

$$S_6(\tilde{\rho}) = \frac{1}{2\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{m J_m(\tilde{\rho}) J'_{\nu_2-\nu_1+m}(\kappa\tilde{\rho})}{\nu_1 - m} + \frac{1}{2\kappa\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa\tilde{\rho}) J'_{\nu_1-\nu_2+m}(\tilde{\rho})}{\nu_2 - m} \tag{9}$$

where  $J_m$  is the Bessel function of order  $m$ , the prime denotes the derivative with respect to the argument of  $J_m$ , and  $\kappa = k_2/k_1$ .  $S_1$  and  $S_3$  are deduced from  $S_2$  and  $S_4$  by changing  $\nu_2$  into  $\nu_1$  and by setting  $\kappa = 1$ . The sums  $S_2$  and  $S_4$  (and thus the sums  $S_1$  and  $S_3$ ) can

be carried out analytically, as well as the sums  $S_5$  and  $S_6$  when  $\kappa = 1$ . However their exact expression will not be reported here as they do not give any useful information about the phenomenon of coherent acceleration described in this paper.

To first order in the wave amplitudes, the original variables  $(I, \theta)$  are related to the new variables  $(\tilde{I}, \tilde{\theta})$  by

$$I = \tilde{I} + \varepsilon_1 \sum_{m=-\infty}^{+\infty} \frac{m J_m(\tilde{\rho}) \cos(m\tilde{\theta} - \nu_1 \tau)}{\nu_1 - m} + \varepsilon_2 \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa \tilde{\rho}) \cos(m\tilde{\theta} - \nu_2 \tau)}{\kappa(\nu_2 - m)} \quad (10)$$

$$\theta = \tilde{\theta} - \frac{\varepsilon_1}{\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{J'_m(\tilde{\rho}) \sin(m\tilde{\theta} - \nu_1 \tau)}{\nu_1 - m} - \frac{\varepsilon_2}{\tilde{\rho}} \sum_{m=-\infty}^{+\infty} \frac{J'_m(\kappa \tilde{\rho}) \sin(m\tilde{\theta} - \nu_2 \tau)}{\nu_2 - m} \quad (11)$$

It is sufficient to calculate the change of variables to first order as the second order terms give negligible contributions. However, we need evaluate the second order contributions to  $\tilde{H}_2^{(off)}$ , as there are no first order terms in  $\tilde{H}_2^{(off)}$ .

For  $\rho < \min(\nu_i - \sqrt{\varepsilon_i})$ , i.e. below the chaotic domain, the change of variables given by (10) is close to identity. This is a necessary condition for the perturbation analysis to give accurate results, and is the reason why, in the rest of the paper, we identify the variations in  $\tilde{I}$  with the variations in  $I$ . The change of variables given by (11) is also close to an identity transformation, except for very small values of  $\rho$ . However, since  $\theta$  appears in Hamilton's equations through  $X = \rho \sin \theta$  and  $\dot{X} = \rho \cos \theta$ , and since the transformation  $(X, \dot{X}) \mapsto (\tilde{X}, \dot{\tilde{X}})$  is approximately an identity transformation, the results of the perturbation analysis are appropriate for describing ion acceleration for any  $\rho < \min(\nu_i - \sqrt{\varepsilon_i})$ .

We now focus on the case when the wave frequencies are above the cyclotron frequency, i.e.  $\nu_1 > 1$  and  $\nu_2 > 1$ . Such a choice is mainly motivated by the results reported in Ref. 10.

Then, a Taylor expansion of  $S_i$ 's to the lowest order in  $\tilde{\rho}$  indicate that, when  $\tilde{\rho} < \min(\nu_1, \nu_2)$ ,  $S_3(\tilde{\rho})$ ,  $S_4(\tilde{\rho})$  and  $S_5(\tilde{\rho})$  are negligible in (5). These results are confirmed numerically. Hence, for  $\tilde{\rho} < \min(\nu_1, \nu_2)$ ,  $\tilde{H}_2^{(off)}$  can be approximated by

$$\tilde{H}_2^{(off)} = \varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho}) + \delta_4 \varepsilon_1 \varepsilon_2 \cos[(\nu_1 - \nu_2)(\theta - \tau) + \varphi_1 - \varphi_2] S_6(\tilde{\rho}). \quad (12)$$

We then define a canonical change of variables  $(\tilde{I}, \tilde{\theta}) \mapsto (\tilde{J}, \tilde{\Phi})$  using the generating function

$$F = \tilde{J}(\tilde{\theta} - \tau) \quad (13)$$

which yields  $\tilde{J} = \tilde{I}$ , and  $\tilde{\Phi} = \tilde{\theta} - \tau$ . In variables  $(\tilde{J}, \tilde{\Phi})$ , (12) is changed into

$$\tilde{H}^{(off)} = \varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho}) + \delta_4 \varepsilon_1 \varepsilon_2 \cos[(\nu_1 - \nu_2)\tilde{\Phi} + \varphi_1 - \varphi_2] S_6(\tilde{\rho}). \quad (14)$$

The Hamiltonian (14) is integrable and the ion orbits are obtained by solving  $\tilde{H}^{(off)} = \text{const}$ . If  $\delta_4 = 0$ , i.e. if  $(\nu_1 - \nu_2)$  is not an integer, then solving  $\tilde{H}^{(off)} = \text{const}$  yields  $\tilde{\rho} = \text{const}$ . In such a case the original action  $I$  only fluctuates by an amount of order  $\varepsilon_1$  or  $\varepsilon_2$ , and behaves as in the case of one wave: there is no acceleration. This is physically clear because when  $(\nu_1 - \nu_2)$  is not an integer, the action of the waves only amounts to some rapid perturbations which do not really affect the ion motion. Conversely, when  $(\nu_1 - \nu_2)$  is an integer, the non-linear beating of the waves gives rise to a slowly varying force acting on the ion and coherently accelerating it. The third term on the right-hand side of (14) provides this force. The amount of energy an ion gets depends on the relative importance of this term

compared to the stabilizing terms  $\varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho})$ . The study of the competition between the accelerating and stabilizing terms, as a function of the wave amplitudes, wavenumbers, frequencies and initial phases, is the main topic of the remaining of this subsection.

If  $(\nu_1 - \nu_2)$  is an integer, there is a large variation in  $\tilde{\rho}$  if  $\varepsilon_1 \varepsilon_2 S_6(\tilde{\rho})$  is large compared to  $\varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho})$ . As  $(\varepsilon_1 \varepsilon_2)/(\varepsilon_1^2 + \varepsilon_2^2)$  is maximum when  $\varepsilon_1 = \varepsilon_2$ , the largest acceleration will occur when the two waves have the same amplitude. This is the case we consider from here on.

It is clear from (14) that the initial phases  $\varphi_1$  and  $\varphi_2$  play no role in the acceleration mechanism since a translation of the angle  $\tilde{\Phi}$  to  $\tilde{\Phi} + (\varphi_1 - \varphi_2)/(\nu_1 - \nu_2)$  eliminates any dependence on initial phases. Hence, without loss of generality, we will consider the case when  $\varphi_1 = \varphi_2 = 0$ . When  $\varepsilon_1 = \varepsilon_2 = \varepsilon$  and  $\varphi_1 = \varphi_2 = 0$ ,  $\tilde{H}^{(off)}$  is

$$\tilde{H}^{(off)} = \varepsilon^2 \left\{ S_1(\tilde{\rho}) + S_2(\tilde{\rho}) + \cos[(\nu_1 - \nu_2)\tilde{\Phi}] S_6(\tilde{\rho}) \right\}. \quad (15)$$

The orbits of  $\tilde{H}$ , obtained from (15), are independent of the wave amplitudes: dividing each amplitude by the same coefficient will not change the amount of energy an ion can gain from the waves. However, decreasing  $\varepsilon$  increases the time needed for acceleration as it is clear from (15) that this time is proportional to  $\varepsilon^{-2}$ .

Let us now study the dependence of the acceleration mechanism on the ratio of the wavenumbers  $\kappa = k_2/k_1$ , and on the integer difference  $(\nu_1 - \nu_2)$ . Solving  $\tilde{H}^{(off)} = \varepsilon^2 \text{const}$  leads to

$$\cos(\nu_1 - \nu_2)\tilde{\Phi} = \frac{\text{const} - S_1(\tilde{\rho}) - S_2(\tilde{\rho})}{S_6(\tilde{\rho})} \quad (16)$$

If the initial value of  $\tilde{\rho}$  lies between two zeros of  $S_6(\tilde{\rho})$ , say  $\tilde{\rho}_1^*$  and  $\tilde{\rho}_2^*$ , then it is clear from (16) that  $\tilde{\rho}$  will always remain between  $\tilde{\rho}_1^*$  and  $\tilde{\rho}_2^*$ . This implies that the amount of energy an ion gains from the waves is a discontinuous function of the ratio  $\kappa$  of the 2 wavenumbers, as illustrated in Fig. 2 in the case when  $\nu_1 - \nu_2 = 1$ . These discontinuities occur when one of the zeros of  $S_6(\tilde{\rho})$  is equal to the initial value  $\tilde{\rho}_0$  of the normalized Larmor radius. For example, when  $\nu_1 - \nu_2 = 1$  and  $\kappa = 0.925$ , the first zero,  $\tilde{\rho}^*$ , of  $S_6(\tilde{\rho})$  is just slightly above  $\tilde{\rho}_0 = 53$ . In this case, the ion Larmor radius cannot increase much, as it has to remain less than  $\tilde{\rho}^*$ . Nevertheless, nothing prevents *a priori* the ion Larmor radius to go down to 0, and one can see in Fig. 2 that the minimum value of the Larmor radius is indeed very close to 0 in this case. Thus, when  $\kappa = 0.925$ , the ion is decelerated. On the contrary, when  $\kappa = 0.924$ , the first zero of  $S_6(\tilde{\rho})$  is slightly below  $\tilde{\rho}_0$  so that the ion's Larmor radius cannot decrease much, because it has to remain larger than  $\tilde{\rho}^*$ . However, it can increase up to the value of the second zero of  $S_6(\tilde{\rho})$ . In this case the ion gets a finite acceleration, and its Larmor radius increases up to  $\tilde{\rho}_{\max} \simeq 75$ .

When  $\kappa \simeq 1.02$ , both  $\tilde{\rho}_{\min}$  and  $\tilde{\rho}_{\max}$  are very close to  $\tilde{\rho}_0$ . This is due to the fact that the first coefficient of the Taylor expansion of  $S_6(\tilde{\rho})$  goes through 0 for  $\kappa \simeq 1 + 2/\nu_1$ , which implies that for such values of  $\kappa$   $S_6(\tilde{\rho})$  becomes very small compared to  $S_1(\tilde{\rho})$  and  $S_2(\tilde{\rho})$ . In this case there is no acceleration.

Finally, one can see in Fig. 2 that for values of  $\kappa$  close to one, there is a peak of large acceleration. This peak is actually not centered about  $\kappa = 1$  but about a value of  $\kappa$  less than 1. For values of  $\kappa$  corresponding to this peak,  $\tilde{\rho}_{\max}$  gets very close to  $\nu_1$ . For large enough values of  $\varepsilon$ , the lower bound of the stochastic region, as estimated from (4), can actually be

lower than the value of  $\tilde{\rho}_{\max}$  predicted from perturbation theory. For such values of  $\varepsilon$ , the phase space is not partitioned into a regular and a chaotic region. Our perturbation analysis accurately describes the ion dynamics until  $\rho = \min(\nu_1 - \sqrt{\varepsilon_i})$ , which allows us to show that an ion can be coherently accelerated up to this value of the Larmor radius. As soon as  $\rho = \min(\nu_1 - \sqrt{\varepsilon_i})$ , the ion dynamics is no longer integrable and cannot be approximated by a perturbation analysis. The ion then gains energy by the classical stochastic acceleration as described in Refs. 1–4.

When  $(\nu_1 - \nu_2) = 2$ , the situation is about the same as when  $\nu_1 - \nu_2 = 1$  except for the positions of the discontinuities of  $\tilde{\rho}_{\max}$  and  $\tilde{\rho}_{\min}$  which do not occur for the same values of  $\kappa$  because the zeros of  $S_6(\tilde{\rho})$  are not the same when  $\nu_1 - \nu_2 = 1$  as when  $\nu_1 - \nu_2 = 2$ . Fig. 3 plots  $\tilde{\rho}_{\max}$  versus  $\kappa$  when  $\tilde{\rho}_0 = 10$  and in both cases :  $\nu_1 = 140.3$  and  $\nu_2 = 139.3$ , and  $\nu_1 = 140.3$  and  $\nu_2 = 138.3$ . For the range of values of  $\kappa$  corresponding to Fig. 3,  $\tilde{\rho}_0$  is not a zero of  $S_6(\tilde{\rho})$ , which explains that  $\tilde{\rho}_{\max}$  is a smooth function of  $\kappa$  except when  $\kappa \simeq 1 + 2/\nu_1$  where  $\tilde{\rho}_{\max}$  drops both when  $\nu_1 - \nu_2 = 1$  and when  $\nu_1 - \nu_2 = 2$ . The drop actually spans a larger range of values of  $\kappa$  when  $\nu_1 - \nu_2 = 2$  than when  $\nu_1 - \nu_2 = 1$  and is due to the fact that the first coefficient of the Taylor expansion of  $S_6(\tilde{\rho})$  is minimum when  $\kappa = 1 + 2\nu_1$  in the case when  $\nu_1 - \nu_2 = 2$ .

There is also a peak of large acceleration when  $\nu_1 - \nu_2 = 2$ , but it is shifted to the left compared to the case when  $\nu_1 - \nu_2 = 1$ . This is a general trend: as  $\nu_1 - \nu_2$  increases, this peak moves to the left.

Finally, it is important to remark that when  $\nu_1 - \nu_2 = 1$  and when  $\nu_1 - \nu_2 = 2$ , the maximum energy an ion reaches is about the same when  $\tilde{\rho}_0 = 10$  as when  $\tilde{\rho}_0 = 53$ . An ion can reach the



chaotic domain of phase space whatever its initial energy. This last feature is only true when  $(\nu_1 - \nu_2) \leq 2$ . When  $\nu_1 - \nu_2 \geq 3$  only ions with initial energies greater than a threshold value will reach high energies. This is due to the fact that  $S_6(\tilde{\rho}) \sim \tilde{\rho}^{(\nu_1 - \nu_2)}$  for small values of  $\tilde{\rho}$  while  $S_1(\tilde{\rho}), S_2(\tilde{\rho}) \sim \tilde{\rho}^2$  independent of  $(\nu_1 - \nu_2)$ . Thus, when  $(\nu_1 - \nu_2) \geq 3$ , for small values of  $\tilde{\rho}$ ,  $S_6(\tilde{\rho})$  becomes very small compared to  $S_1(\tilde{\rho})$  and  $S_2(\tilde{\rho})$  and there is no acceleration. As  $\tilde{\rho}_0$  decreases,  $|\kappa - 1|$  has to increase in order for the ions to get accelerated. The maximum value of the Larmor radius is still limited by the zeros of  $S_6(\tilde{\rho})$ . The first zero of  $S_6(\tilde{\rho})$  moves to lower values of  $\rho$  as  $|\kappa - 1|$  is increased, thus the maximum value that  $\rho$  can attain decreases as  $\tilde{\rho}_0$  decreases (see Fig.4).

In conclusion, we have shown that an ion, regardless of its initial energy, may be accelerated by two waves. Moreover, in an experiment where the wave characteristics can be specified, one can have, with two waves, a lot of control on the ion dynamics below the stochastic region. Indeed, by changing the ratio of the two wavenumbers or the difference between the two waves frequencies, one can choose the amount of energy an ion gains from the waves as well as the range of initial energy of the ions that are accelerated.

## B. Coherent acceleration in the case of two on-resonance waves

We now study the dynamics defined by (2), in the case of two waves with both  $\nu_1 = n_1$  and  $\nu_2 = n_2$  being integers. We will restrict ourselves to the case where  $\rho$  is less than  $\min(n_1, n_2)$ . Thus, we again focus on a region of phase space where the motion is well described by an integrable Hamiltonian. As in the previous section, this Hamiltonian is derived

from  $H$  by using perturbation theory up to second order in the wave amplitudes. We then obtain the following transformed Hamiltonian:

$$\begin{aligned}
\tilde{H}_2^{(on)} = & \varepsilon_1 J_{n_1}(\tilde{\rho}) \cos[n_1(\tilde{\theta} - \tau) + \varphi_1] + \varepsilon_2 \frac{J_{n_2}(\kappa\tilde{\rho})}{\kappa} \cos[n_2(\tilde{\theta} - \tau) + \varphi_2] \\
& + \varepsilon_1^2 \bar{S}_1(\tilde{\rho}) + \varepsilon_2^2 \bar{S}_2(\tilde{\rho}) + \varepsilon_1^2 \cos[2n_1(\tilde{\theta} - \tau) + 2\varphi_1] \bar{S}_3(\tilde{\rho}) \\
& + \varepsilon_2^2 \cos[2n_2(\tilde{\theta} - \tau) + 2\varphi_2] \bar{S}_4(\tilde{\rho}) \\
& + \varepsilon_1 \varepsilon_2 \cos[(n_1 + n_2)(\tilde{\theta} - \tau) + \varphi_1 + \varphi_2] \bar{S}_5(\tilde{\rho}) \\
& + \varepsilon_1 \varepsilon_2 \cos[(n_1 - n_2)(\tilde{\theta} - \tau) + \varphi_1 - \varphi_2] \bar{S}_6(\tilde{\rho})
\end{aligned} \tag{17}$$

where

$$\bar{S}_2(\tilde{\rho}) = \frac{1}{2\kappa\tilde{\rho}} \sum_{m \neq n_2} \frac{m J_m(\kappa\tilde{\rho}) J'_m(\kappa\tilde{\rho})}{n_2 - m} \tag{18}$$

$$\bar{S}_4(\tilde{\rho}) = \frac{1}{2\kappa\tilde{\rho}} \sum_{m \neq n_2} \frac{m J_m(\kappa\tilde{\rho}) J'_{2n_2-m}(\kappa\tilde{\rho})}{n_2 - m} \tag{19}$$

$$\bar{S}_5(\tilde{\rho}) = \frac{1}{2\tilde{\rho}} \sum_{m \neq n_1} \frac{m J_m(\tilde{\rho}) J'_{n_1+n_2-m}(\kappa\tilde{\rho})}{n_1 - m} + \frac{1}{2\kappa\tilde{\rho}} \sum_{m \neq n_2} \frac{m J_m(\kappa\tilde{\rho}) J'_{n_1+n_2-m}(\tilde{\rho})}{n_2 - m} \tag{20}$$

$$\bar{S}_6(\tilde{\rho}) = \frac{1}{2\tilde{\rho}} \sum_{m \neq n_1} \frac{m J_m(\tilde{\rho}) J'_{n_2-n_1+m}(\kappa\tilde{\rho})}{n_1 - m} + \frac{1}{2\kappa\tilde{\rho}} \sum_{m \neq n_2} \frac{m J_m(\kappa\tilde{\rho}) J'_{n_1-n_2+m}(\tilde{\rho})}{n_2 - m} \tag{21}$$

$\bar{S}_1$  and  $\bar{S}_3$  are obtained from  $\bar{S}_2$  and  $\bar{S}_4$ , respectively, by replacing  $n_2$  by  $n_1$ , and by setting  $\kappa = 1$ .

To first order in the wave amplitudes, the original variables  $(I, \theta)$  are related to the new variables  $(\tilde{I}, \tilde{\theta})$  by

$$I = \tilde{I} + \varepsilon_1 \sum_{m \neq n_1} \frac{m J_m(\tilde{\rho}) \cos(m\tilde{\theta} - n_1\tau)}{n_1 - m} + \varepsilon_2 \sum_{m \neq n_2} \frac{m J_m(\kappa\tilde{\rho}) \cos(m\tilde{\theta} - n_2\tau)}{\kappa(n_2 - m)} \tag{22}$$

$$\theta = \tilde{\theta} - \frac{\varepsilon_1}{\tilde{\rho}} \sum_{m \neq n_1} \frac{J'_m(\tilde{\rho}) \sin(m\tilde{\theta} - n_1\tau)}{n_1 - m} - \frac{\varepsilon_2}{\tilde{\rho}} \sum_{m \neq n_2} \frac{J'_m(\kappa\tilde{\rho}) \sin(m\tilde{\theta} - n_2\tau)}{n_2 - m} \quad (23)$$

As in the off-resonance case, it can be shown that the only sums that give a non-negligible contribution to  $\tilde{H}_2^{(on)}$  are  $\tilde{S}_1$ ,  $\tilde{S}_2$ , and  $\tilde{S}_6$ . Moreover, for  $\tilde{\rho} \leq \min(n_1, n_2)$ ,  $\tilde{S}_i(\tilde{\rho}) \simeq S_i(\tilde{\rho})$  for  $i = 1, 2, 6$ . Indeed, because of the rapid decrease of  $J_n(x)$  when  $x$  becomes less than  $n$ , the only significant terms in the  $S_i$ 's and the  $\tilde{S}_i$ 's are those which have Bessel functions of order  $m$  such that  $m \leq \tilde{\rho}$ . Hence, these are the same terms for  $S_i$  and  $\tilde{S}_i$ , and for such values of  $m$ ,  $n_1 - m \simeq \nu_1 - m$  and  $n_2 - m \simeq \nu_2 - m$  provided that  $n_1 = \text{Int}(\nu_1)$  and  $n_2 = \text{Int}(\nu_2)$ , where  $\text{Int}(\nu)$  stands for the integer part of  $\nu$ . Moreover, numerically calculating  $\tilde{S}_i(\tilde{\rho})$  and  $S_i(\tilde{\rho})$  shows that when  $\tilde{\rho} \leq \min(n_1, n_2)$ ,  $\tilde{S}_i(\tilde{\rho}) \simeq S_i(\tilde{\rho})$ . Therefore, for values of  $\tilde{\rho}$  such that  $\tilde{\rho} \leq \min(n_1, n_2)$

$$\tilde{H}_2^{(on)} \simeq \varepsilon_1 J_{n_1}(\tilde{\rho}) \cos[n_1(\tilde{\theta} - \tau) + \varphi_1] + \varepsilon_2 \frac{J_{n_2}(\kappa\tilde{\rho})}{\kappa} \cos[n_2(\tilde{\theta} - \tau) + \varphi_2] + \tilde{H}_2^{(off)} \quad (24)$$

It is clear from (24) that the only difference between the on and off-resonance cases comes from the presence of a first order term in  $\tilde{H}_2^{(on)}$  which does not exist in  $\tilde{H}_2^{(off)}$ . If  $|n_1 - n_2| < \min(n_1, n_2)$  (which is the only case we consider here), when  $\tilde{\rho} \rightarrow 0$ , this first order term decreases more rapidly than  $\tilde{H}_2^{(off)}$ . This implies that there exists a value  $\tilde{\rho}_l(\varepsilon_1, \varepsilon_2)$  such that for  $\tilde{\rho} \leq \tilde{\rho}_l(\varepsilon_1, \varepsilon_2)$ ,  $\tilde{H}_2^{(on)} \simeq \tilde{H}_2^{(off)}$ . In other words, in the region of phase space corresponding to  $\tilde{\rho} \leq \tilde{\rho}_l$  the orbits for two on-resonance waves are very close to the orbits for two off-resonance waves. In particular, if for a given initial condition the maximal value of the Larmor radius of an ion in two off-resonance waves is less than  $\tilde{\rho}_l$ , then the Larmor radius of an ion in two on-resonance waves will never exceed  $\tilde{\rho}_l$  either. Therefore, the figures 2 to

4 are also valid for two on-resonance waves, except for values of  $\kappa$  where  $\tilde{\rho}_{max}$  is predicted to be larger than  $\tilde{\rho}_l$ . For such values of  $\kappa$  the first order terms in  $\tilde{H}_2^{(on)}$  have to be taken into account as  $\tilde{\rho}$  can reach the lower bound of the stochastic phase space  $\min(n_i - \sqrt{\varepsilon_i})$ . This is illustrated in Fig. 5 where  $\tilde{\rho}_{max}$  is plotted as a function of  $\kappa$  for  $n_1 = 140$  and  $n_2 = 139$ . For values of  $\kappa$  less than 0.978 or larger than 1.001,  $\tilde{\rho}_{max}$  evolves smoothly with  $\kappa$  and assumes the same values as the ones reported in Fig. 2 for two off-resonance waves. Two discontinuities occur at  $\kappa \simeq 0.978$  and  $\kappa \simeq 1.001$  due to the effects of the first order terms in  $\tilde{H}_2^{(on)}$  which bring  $\tilde{\rho}_{max}$  to values larger than  $\min(n_1, n_2)$ . Our perturbation analysis is only valid for  $\tilde{\rho} < \min(n_i - \sqrt{\varepsilon_i})$ . Thus, the physical situation is as follows: the ion is coherently accelerated until  $\rho = \min(n_i - \sqrt{\varepsilon_i})$ , after which the ion experiences a stochastic acceleration. Hence, the effect of the first order terms is to enhance the acceleration by providing an easier access to the stochastic region.

It is clear that, for smaller wave amplitudes, the first order terms will be more important in  $H_2^{(on)}$  and, hence, will have a significant effect on a larger part of phase space. This implies that  $\tilde{\rho}_l(\varepsilon_1, \varepsilon_2)$  is an increasing function of the wave amplitudes. Consequently, as the wave amplitudes are lowered, a larger fraction of ions can experience stochastic acceleration. This is the same kind of effect as the one described in Ref. 11 for one on-resonance wave. However, for high harmonics,  $\tilde{\rho}_l$  is quite insensitive to the value of the wave amplitudes and is close to  $\min(n_1, n_2)$ , as can be seen in Fig. 5.

### C. Relaxation of the condition on the wave frequencies for acceleration

Sections IV.A and IV.B showed that an ion can be accelerated by two waves in a magnetic field provided that the difference in the wave frequencies is an integer multiple of the cyclotron frequency. We show here that the condition on the wave frequencies can actually be relaxed and that there can be some acceleration even if

$$\nu_1 - \nu_2 = n + \delta\nu \quad (25)$$

where  $n$  is an integer, and  $|\delta\nu| \ll 1$  is a small number scaling as  $\varepsilon_1\varepsilon_2$ . Indeed, it is always possible to define a canonical change of variables in such a way as the term involving  $\cos[(\nu_1 - \nu_2)\tau]$  is taken into account in  $\tilde{H}_2^{(off)}$ , whatever the value of  $(\nu_1 - \nu_2)$ . In the case where  $\varphi_1 = \varphi_2$ , the Hamiltonian thus obtained is

$$\tilde{H} = \tilde{I} + \varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho}) + \varepsilon_1\varepsilon_2 \cos[n(\tilde{\theta} - \tau) - \delta\nu\tau] S_6(\tilde{\rho}), \quad (26)$$

$S_6(\tilde{\rho})$  being defined by (9) with  $(\nu_1 - \nu_2)$  replaced by  $n$ . Moreover, it can be shown mathematically that there exists a  $\delta\nu^*$ , scaling as  $\varepsilon_1\varepsilon_2$ , such that when  $\delta\nu \leq \delta\nu^*$ , if the last term of (26) is omitted, then this Hamiltonian becomes irrelevant to describe the dynamics defined by (2). Indeed, if the last term of (26) is omitted then, when  $\delta\nu \leq \delta\nu^*$ , the change of variables  $(I, \theta) \rightarrow (\tilde{I}, \tilde{\theta})$  is no longer one-to-one, and the terms of order larger than 2 are no longer negligible in the perturbation series. However, such complicated mathematical developments will not be reported as they are not essential to understanding the acceleration when wave

frequencies are not exactly separated by an integer multiple of the ion cyclotron frequency. Indeed, let us first notice that the Hamiltonian (26) is integrable. This can be easily seen by performing the canonical change of variables  $(\tilde{I}, \tilde{\theta}) \rightarrow (\tilde{K}, \tilde{\psi})$  using the generating function

$$G = \tilde{K}(\tilde{\theta} - \tau - \delta\nu\tau/n) \quad (27)$$

which yields  $\tilde{K} = \tilde{I}$ , and  $\tilde{\psi} = \tilde{\theta} - \tau - \delta\nu/N\tau$ . In the variables  $(\tilde{K}, \tilde{\psi})$ , (26) becomes

$$\tilde{H} = -\frac{\delta\nu}{n}\tilde{K} + \varepsilon_1^2 S_1(\tilde{\rho}) + \varepsilon_2^2 S_2(\tilde{\rho}) + \varepsilon_1 \varepsilon_2 \cos(n\tilde{\psi}) S_6(\tilde{\rho}) \quad (28)$$

which is time independent. Hence, the orbits of (28) are obtained by solving  $\tilde{H} = \text{const.}$  When solving this equation, it is clear that if there is some acceleration when  $\delta\nu = 0$ , this acceleration persists when  $\delta\nu \neq 0$  only if  $\varepsilon_1 \varepsilon_2 S_6(\tilde{\rho})$  is at least of the order of  $(\delta\nu/n)\tilde{I}$ . This shows that  $\delta\nu$  must scale as  $\varepsilon_1 \varepsilon_2$  for acceleration to take place. When this condition is fulfilled, it is then clear that the amount of energy gained by an ion is of the same order of magnitude as in the case when  $\delta\nu = 0$ . Hence, the results derived in the sections IV.A and IV.B remain relevant. In particular, the acceleration will be all the more important as the amplitude of  $S_6(\tilde{\rho})$  is large, and the curve plotting the maximum and minimum values of  $\tilde{\rho}$  as a function of  $k_2/k_1$  also has some discontinuities, due to the existence of zeros in  $S_6(\tilde{\rho})$ , as in Fig. 2. Therefore, the results obtained in sections IV.A and IV.B describe in a very complete way the acceleration of low energy ions by two electrostatic waves propagating perpendicularly to a uniform magnetic field.

## D. Acceleration of low energy ions in an arbitrary discrete wave spectrum

In this section we study the dynamics defined by (2) in the case when an arbitrary number of on and off-resonance waves are included. The study is still restricted to the part of phase space which is below the stochastic region, namely we only consider values of  $\rho$  such that  $\rho \leq \min(\nu_i)$ . We show here that the results obtained in the case of two waves readily apply to the case of many waves. This occurs simply because a perturbation theory led up to the second order in the wave amplitudes only involves the non-linear interaction of pairs of waves. Thus, to second order, and using the same approximations as before, the Hamiltonian (2) is transformed to

$$\begin{aligned} \tilde{H} \simeq & \sum_{\nu_i \in \mathcal{N}} \frac{\varepsilon_i}{\kappa_i} J_{\nu_i}(\kappa_i \tilde{\rho}) \cos(\nu_i \tilde{\Phi} + \varphi_i) \\ & + \sum_{i=1}^N \varepsilon_i^2 S_2^{(i)}(\tilde{\rho}) + \sum_{(\nu_i - \nu_j) \in \mathcal{N}^*} \varepsilon_i \varepsilon_j S_6^{(i,j)}(\tilde{\rho}) \cos[(\nu_i - \nu_j) \tilde{\Phi} + \varphi_i - \varphi_j] \end{aligned} \quad (29)$$

where  $\mathcal{N}$  denotes the set of positive integers, and  $\mathcal{N}^*$  the set of strictly positive integers,  $\mathcal{N}^* = \mathcal{N} \setminus \{0\}$ . If  $\nu_i$  is not an integer,  $S_2^{(i)}$  is given by (6) with  $\kappa$  replaced by  $\kappa_i$  and  $\nu_2$  replaced by  $\nu_i$ ; if  $\nu_i$  is an integer,  $S_2^{(i)}$  is given by (18) with the same replacements. As already noted in the case of two waves, the fact that  $\nu_i$  is an integer or not does not really affect the value of  $S_2^{(i)}$ . As for  $S_6^{(i,j)}$  it is defined by

$$S_6^{(i,j)} = \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa_j \tilde{\rho}) J'_{\nu_j - \nu_j + m}(\kappa_j \tilde{\rho})}{\kappa_j (\nu_j - m)} + \sum_{m=-\infty}^{+\infty} \frac{m J_m(\kappa_i \tilde{\rho}) J'_{\nu_j - \nu_i + m}(\kappa_i \tilde{\rho})}{\kappa_i (\nu_i - m)} \quad (30)$$

when  $\nu_i$  and  $\nu_j$  are not integers. If they are integers the term involving  $m = \nu_i$  has to be excluded in the first sum of (30), and the term  $m = \nu_j$  has to be excluded in the second sum of (30). Again, as noted when studying the case of two waves, the fact that  $\nu_i$  and  $\nu_j$  are integers or not does not really affect the values of  $S_6^{(i,j)}(\bar{\rho})$  as long as  $\bar{\rho} \leq \min(\nu_i, \nu_j)$ .

Now, the dependence of the acceleration mechanism on the wave spectrum parameters can be easily deduced from the case of two waves. In particular, the first order terms only play a role in a part of phase space close to the chaotic domain and can be neglected in a first approximation. Therefore, there is acceleration when the sum,  $S_a$ , of the terms  $\varepsilon_i \varepsilon_j S_6^{(i,j)}(\bar{\rho}) \cos[(\nu_i - \nu_j)\Phi + \varphi_i - \varphi_j]$  is at least of the same order as the sum,  $S_s$ , of the stabilizing terms  $\varepsilon_i^2 S_2^i(\bar{\rho})$ . This yields the conditions on the different parameters of the problem in order to obtain acceleration.

In the case where the wave amplitudes are all about the same, the condition on the frequencies is simply that the number of pairs of frequencies separated by an integer multiple of the cyclotron frequency must be at least of the order of the total number of waves. Indeed, otherwise the number of stabilizing terms is much larger than the number of accelerating terms, and there is no acceleration.

As for the amplitudes, we can show, as in section IV.A, that the energization is maximum when two waves whose frequencies are separated by an integer multiple of the cyclotron frequency have the same amplitudes. Moreover, if all the waves have the same normalized amplitude  $\varepsilon$ , then the orbits do not depend on  $\varepsilon$ , except when the first order terms are not negligible, in which case we find that an ion accesses more easily the chaotic domain of phase space as  $\varepsilon$  is decreased. The time needed for an ion to be accelerated scales, as in the case



of two waves, as  $\varepsilon^{-2}$ .

The main difference with the case of two waves comes from the fact that changing the values of the initial phases  $\varphi_i$  does not only result in a shift of the orbit. The orbit explicitly depends on the choice of the  $\varphi_i$ 's as long as there is more than one term in the sum,  $S_a$ , of the accelerating terms. When the number of terms in  $S_a$  is of the order of unity, then the amount of energy gained by an ion is about the same whatever the phases. Nevertheless, if the number  $N_a$  of terms in  $S_a$  is large, then this sum scales differently with  $N_a$  depending on the choice of the phases  $\varphi_i$ 's. For example, if the  $\varphi_i$ 's are all the same then the sum will scale as  $N_a$ , while if the  $\varphi_i$ 's are chosen randomly the sum will scale as  $\sqrt{N_a}$ . Hence, for a large number of waves, one expects to obtain more acceleration when the initial phases are coherent than when they are random. Numerical results show that there is very little acceleration when the initial phases of the waves are randomly distributed.

Finally, the way the gain of energy varies with the wavenumbers can also be very easily deduced from the study performed for two waves. Indeed, for each pair of waves yielding an acceleration term, one knows how much energy an ion would gain if only these two waves existed. Then, the order of magnitude of the energy an ion gains from the whole spectrum is of the order of the sum of the energies gained from each pair of waves divided by half of the total number of waves. Therefore,  $\bar{\rho}_{\min}$  and  $\bar{\rho}_{\max}$  are expected to behave with the wavenumbers in a similar way as in the case of two waves. In order to test this point numerically without having to specify a dispersion relation, we consider the case when the wavenumbers are all about the same value. Then, it is valid to make a first order Taylor expansion of the dispersion relation  $k(\omega)$  and to relate the wavenumbers through a linear

relation  $k_i = k_1 + \alpha(\omega_i - \omega_1)$ , where  $\alpha$  is the inverse of the group velocity evaluated at  $k = k_1$ . Numerically, we fix the values of the wave frequencies and compute the amount of energy gained by an ion when  $\alpha$  is varied. Fig. 6 plots  $\tilde{\rho}_{\min}$  and  $\tilde{\rho}_{\max}$  versus  $\kappa = k_2/k_1$  in the case of multiple waves. This figure is similar to Fig. 3 plotted in the case of two waves. One can notice however that in the case of multiple waves the maximum value of  $\tilde{\rho}_{\max}$  is somewhat lower than in the case of two waves. Yet, in the case of Fig. 6, four pairs of waves give rise to acceleration while there are five waves. Therefore, the ratio of the number of terms giving rise to acceleration divided by the total number of waves is higher than in the case of two waves. The fact that the maximum acceleration is not as high as for two waves simply comes from the fact that the zones of high acceleration do not correspond to the same value of the ratio of the wavenumbers for each pair of waves.

In deriving the Hamiltonian (29) we did not take into account the case when the difference  $(\nu_l - \nu_m)$  between two normalized wave frequencies is so close to an integer that the non-linear interaction of these waves could give rise to an accelerating term even though  $(\nu_l - \nu_m)$  is not an exact integer. If we denote  $\nu_l - \nu_m = n_l + \delta\nu_l$  then, as in the case of two waves, the non-linear interaction of the waves  $l$  and  $m$  has to be included in (29) for a small enough value of  $\delta\nu_l$ , scaling as  $\varepsilon_l \varepsilon_m$ . Including the corresponding term in (29) yields

$$\begin{aligned} \tilde{H} \simeq & \sum_{\nu_i \in \mathcal{N}} \frac{\varepsilon_i}{\kappa_i} J_{\nu_i}(\kappa_i \tilde{\rho}) \cos(\nu_i \tilde{\Phi} + \varphi_i) + \varepsilon_l \varepsilon_m S_6^{(l,m)}(\tilde{\rho}) \cos(n_l \tilde{\Phi} - \delta\nu_l \tau + \varphi_l - \varphi_m) \\ & + \sum_{i=1}^N \varepsilon_i^2 S_2^{(i)}(\tilde{\rho}) + \sum_{(\nu_i - \nu_j) \in \mathcal{N}^*} \varepsilon_i \varepsilon_j S_6^{(i,j)}(\tilde{\rho}) \cos[(\nu_i - \nu_j) \tilde{\Phi} + \varphi_i - \varphi_j] \end{aligned} \quad (31)$$

The Hamiltonian (31) is not integrable, as soon as there is more than one term in the sum

$S_a$ . Therefore, the orbits of (31) cannot be exactly calculated. Nevertheless, during a time  $\tau_0 \sim 1/\delta\nu_l$ , an orbit of (31) is very close to the orbit obtained by setting  $\delta\nu_l = 0$  and solving  $\tilde{H} = \text{const}$ . Since the maximum value of  $\delta\nu_l$  scales as  $\varepsilon_l \varepsilon_m$ , the time  $1/\delta\nu_l$  is at least of the order of the time which gives the coherent acceleration when either  $\delta\nu_l = 0$ , or the term due to the waves  $l$  and  $m$  is negligible. This is actually intuitively obvious, because if  $1/\delta\nu_l$  were much smaller than the time of coherent acceleration, then the term  $\cos(n_l \tilde{\Phi} - \delta\nu_l \tau + \varphi_l - \varphi_m)$  would behave as a fast perturbation and would henceforth be removable using perturbation theory, which is not the case. Therefore, an orbit of  $\tilde{H}$  is close to the orbit obtained by setting  $\delta\nu_l = 0$  during a time  $\tau_0$  larger than the time needed to energize an ion when  $\delta\nu_l = 0$ . This implies that the analysis made on a Hamiltonian like (29) gives the order of magnitude of the energy gained by an ion also in the case of a Hamiltonian of the form (31).

In conclusion, we have shown that the results found in the case of two waves in the sections IV.A and IV.B can be readily generalized to the case of an arbitrary discrete wave spectrum. This shows the universality of the results found in the case of two waves, and implies that the main features of the dynamics of an ion in a discrete spectrum of waves propagating perpendicularly to a uniform magnetic field are known in a complete way, as long as the ion's orbit remains below the stochastic region.

## V. CONCLUSIONS

This paper shows the ability of electrostatic waves, propagating across a uniform magnetic field, to accelerate ions regardless of how small the ions initial energies are. The acceleration

of low-energy ions is coherent, so that there is no threshold for the electric field amplitudes of the waves for the acceleration to take place. The stochasticity threshold that is usually studied in wave-particle interactions is not relevant for this case. The time needed to accelerate low-energy ions is shown to scale inversely as the square of the wave amplitudes. Moreover, we showed that one could have a lot of control on the ions dynamics with electrostatic waves in a magnetized plasma since the maximum energy that the ions can achieve as well as the ion population that is energized can be determined by appropriately choosing the wave frequencies and wavenumbers.

All these results can be readily applied to any physical situation involving coherent wave-particle interaction in a magnetic field. In particular, they prove to be relevant in explaining recent results<sup>10</sup> regarding the acceleration of the ions  $O^+$  and  $H^+$  in the ionosphere.<sup>9</sup>

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## FIGURE CAPTIONS

Fig. 1: Solid line: Poincaré section of the dynamics defined by (2) for 6 waves such that  $\nu_1 = 30, \nu_2 = 29, \nu_3 = 31.75, \nu_4 = 31.25, \nu_5 = 30.5, \nu_6 = 29.25; \kappa_1 = 1, \kappa_2 = 0.98, \kappa_3 = 1, \kappa_4 = 1, \kappa_5 = 1, \kappa_6 = 0.98; \varepsilon_i = 1.8$  and  $\varphi_i = 0$  for  $1 \leq i \leq 6$ . The initial condition is  $\rho_0 = 5, \Phi_0 = \pi/2$ . Dashed line: Orbit obtained after second order perturbation theory for the same parameters as the solid line.

Fig. 2: Maximum and minimum values of the Larmor radius of the orbits of  $\tilde{H}^{(off)}$  corresponding to the initial condition  $\tilde{\rho}_0 = 53, \tilde{\Phi}_0 = 1.57$ , versus  $\kappa$  for 2 waves such that  $\nu_1 = 140.3$  and  $\nu_2 = 139.3$ . The dashed line is  $\tilde{\rho} = 53$ , the first dotted line indicates the largest value of  $\kappa$  below 1 such that  $S_6(53) = 0$ , the second dotted line is at  $\kappa = 1/(2\nu_1)$ , and the third dotted line indicates the lowest value of  $\kappa$  larger than 1 such that  $S_6(53) = 0$ .

Fig. 3: Maximum values of the Larmor radius of the orbits of  $\tilde{H}_2^{(off)}$  corresponding to the initial condition  $\tilde{\rho}_0 = 10, \tilde{\Phi}_0 = 0.4$ , versus  $\kappa$ , for 2 waves such that  $\nu_1 = 140.3$  and  $\nu_2 = 138.3$  (solid line) and  $\nu_1 = 140.3$  and  $\nu_2 = 139.3$  (dashed line). The straight dashed line is at  $\tilde{\rho} = 10$ .

Fig. 4: Maximum values of the Larmor radius of the orbits of  $\tilde{H}_2^{(off)}$  corresponding to the initial condition  $\tilde{\rho}_0 = 20, \tilde{\Phi}_0 = 0.4$ , versus  $\kappa$ , for 2 waves such that  $\nu_1 = 140.3$  and  $\nu_2 = 130.3$ .

Fig. 5: Maximum values of the Larmor radius of the orbits of  $\tilde{H}_2^{(on)}$  corresponding to the

initial condition  $\tilde{\rho}_0 = 50$ ,  $\tilde{\Phi}_0 = \pi/2$ , versus  $\kappa$ , for 2 waves such that  $n_1 = 140$  and  $n_2 = 139$  and  $\varepsilon_1 = \varepsilon_2 = 15$ . The dashed line indicates the lower bound of the stochastic region.

Fig. 6: Maximum and minimum values of the Larmor radius of the orbits of  $\tilde{H}$  corresponding to the initial condition  $\tilde{\rho}_0 = 10$ ,  $\tilde{\Phi}_0 = 0.2$ , versus  $k_2/k_1$  for 5 waves such that  $\nu_1 = 141$  and  $\nu_2 = 140$ ,  $\nu_3 = 139$ ,  $\nu_4 = 143.5$ ,  $\nu_5 = 140.5$ , having all the same amplitudes and the same initial phases. The dashed line is at  $\tilde{\rho} = 10$ .



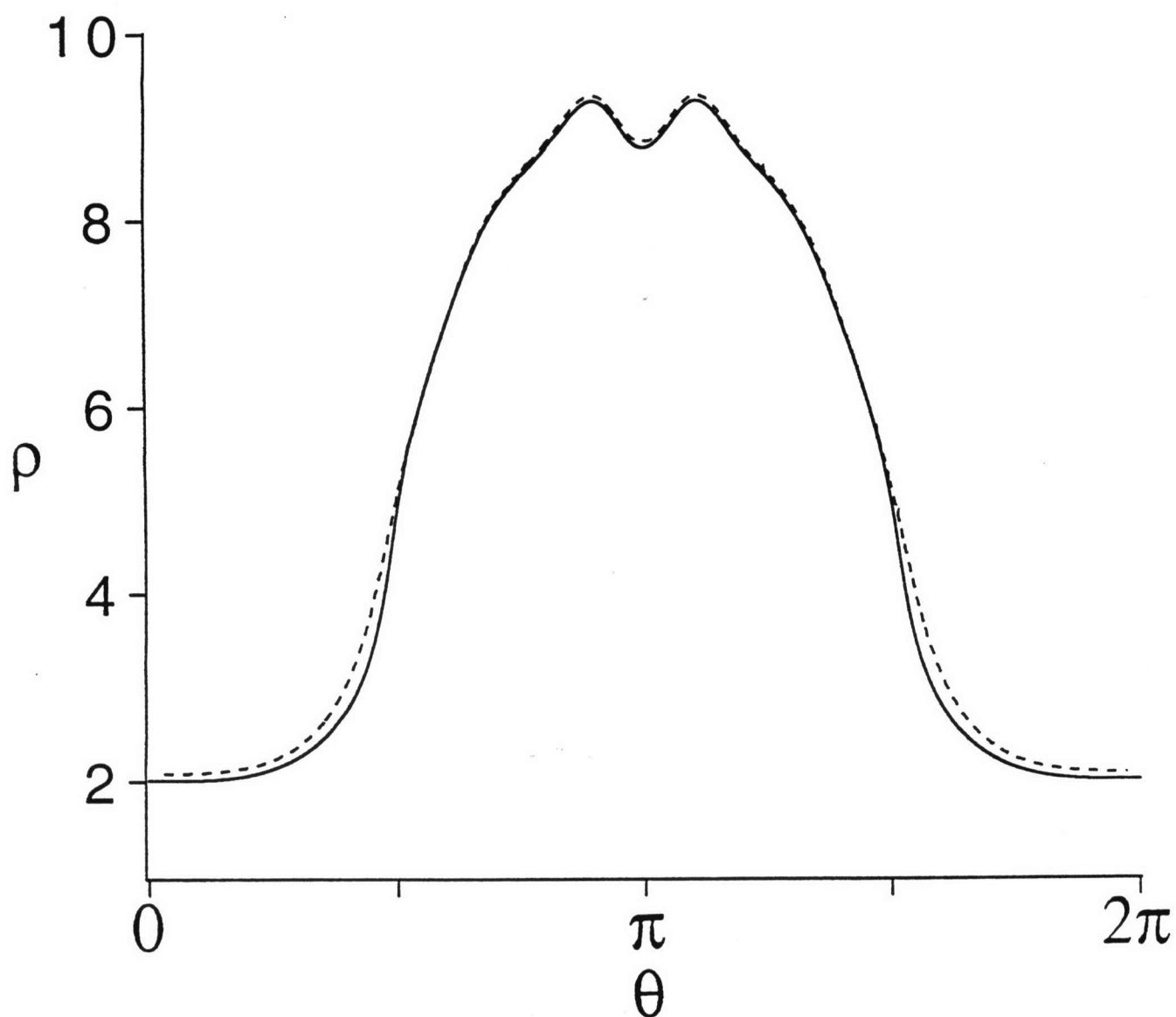


FIGURE 1 Benisti et al., "Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent Acceleration." Physics of Plasmas

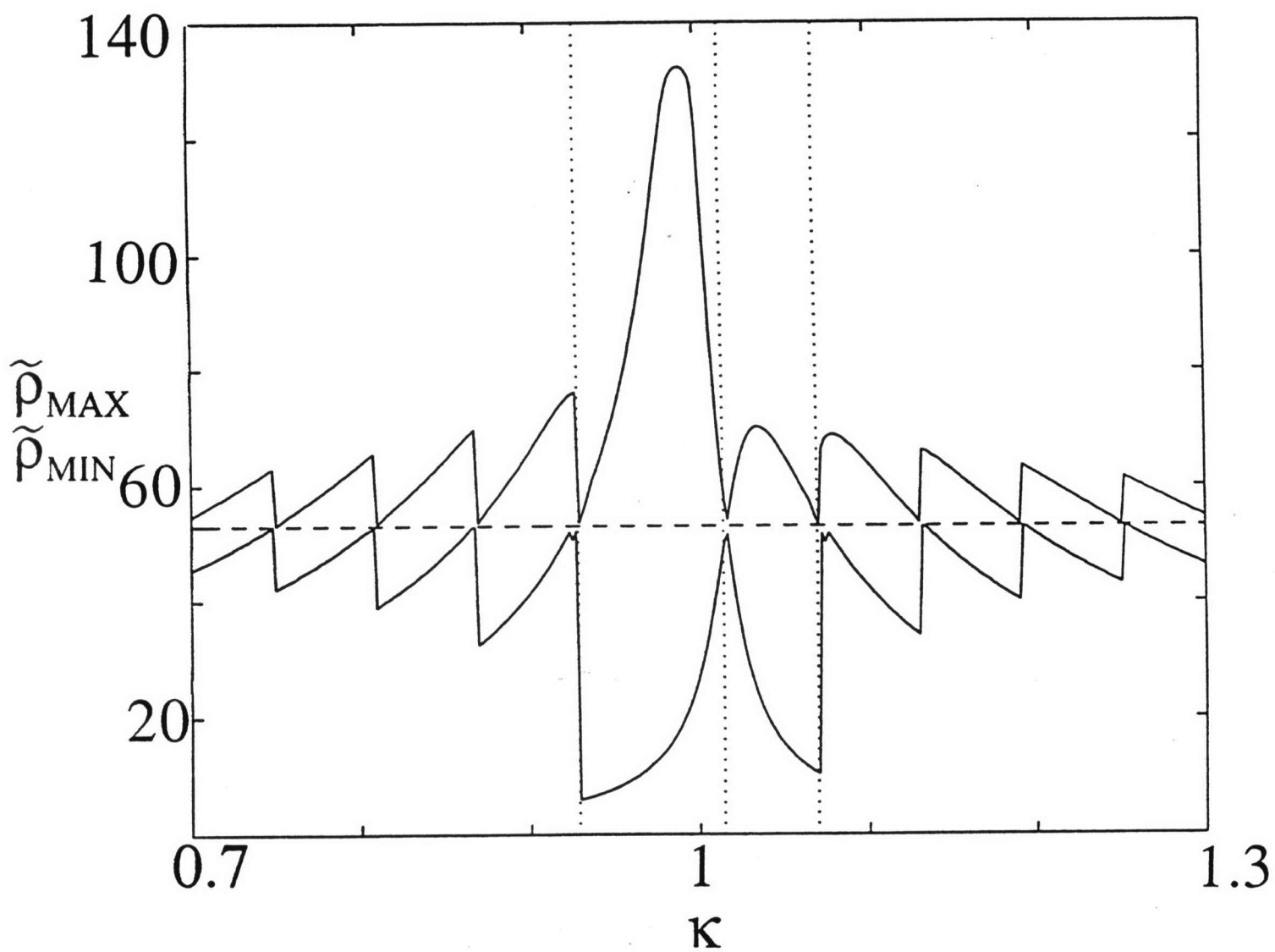


FIGURE 2 Benisti et al., "Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent Acceleration." Physics of Plasmas

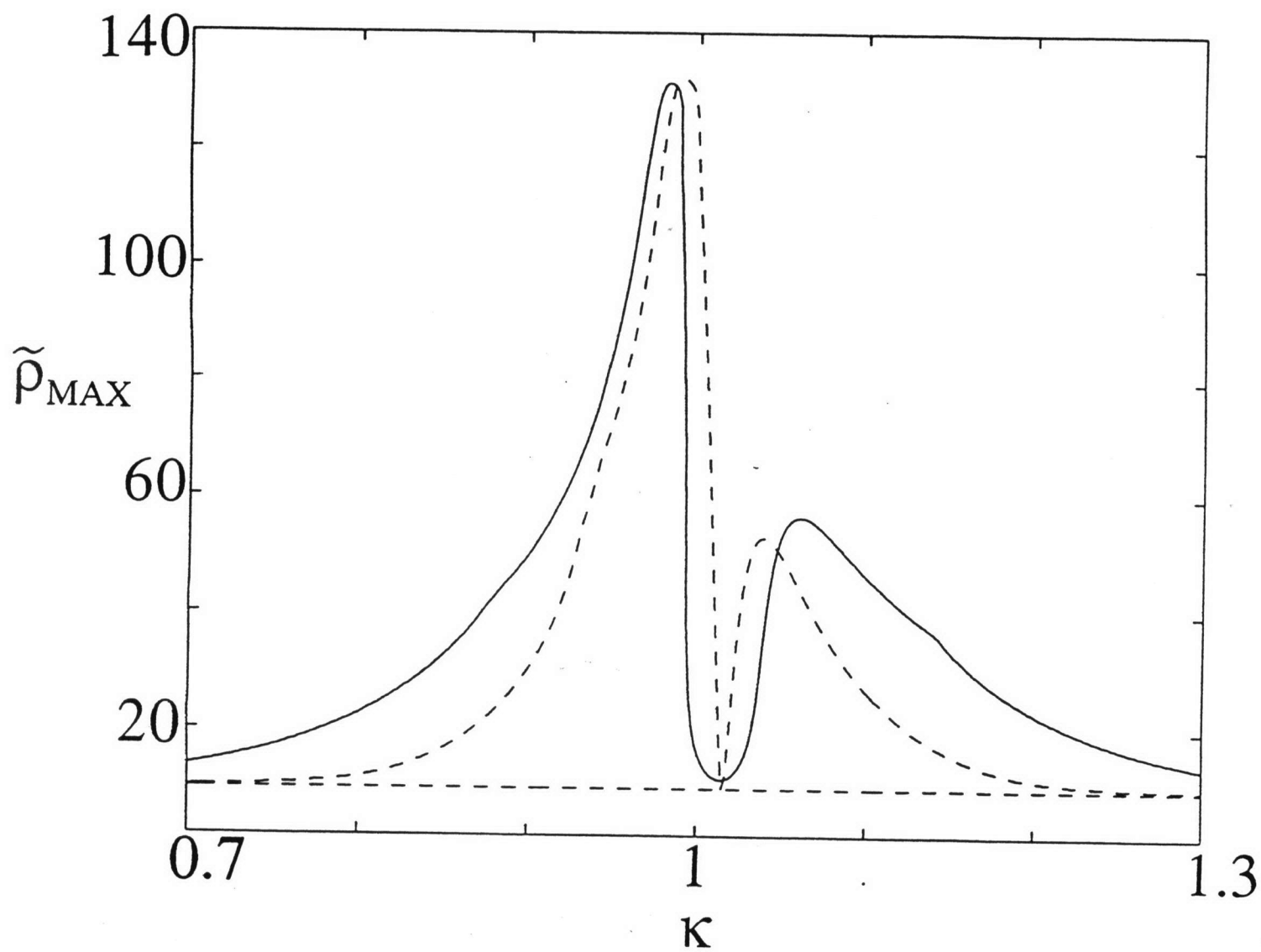


FIGURE 3 Benisti et al., "Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent Acceleration." Physics of Plasmas

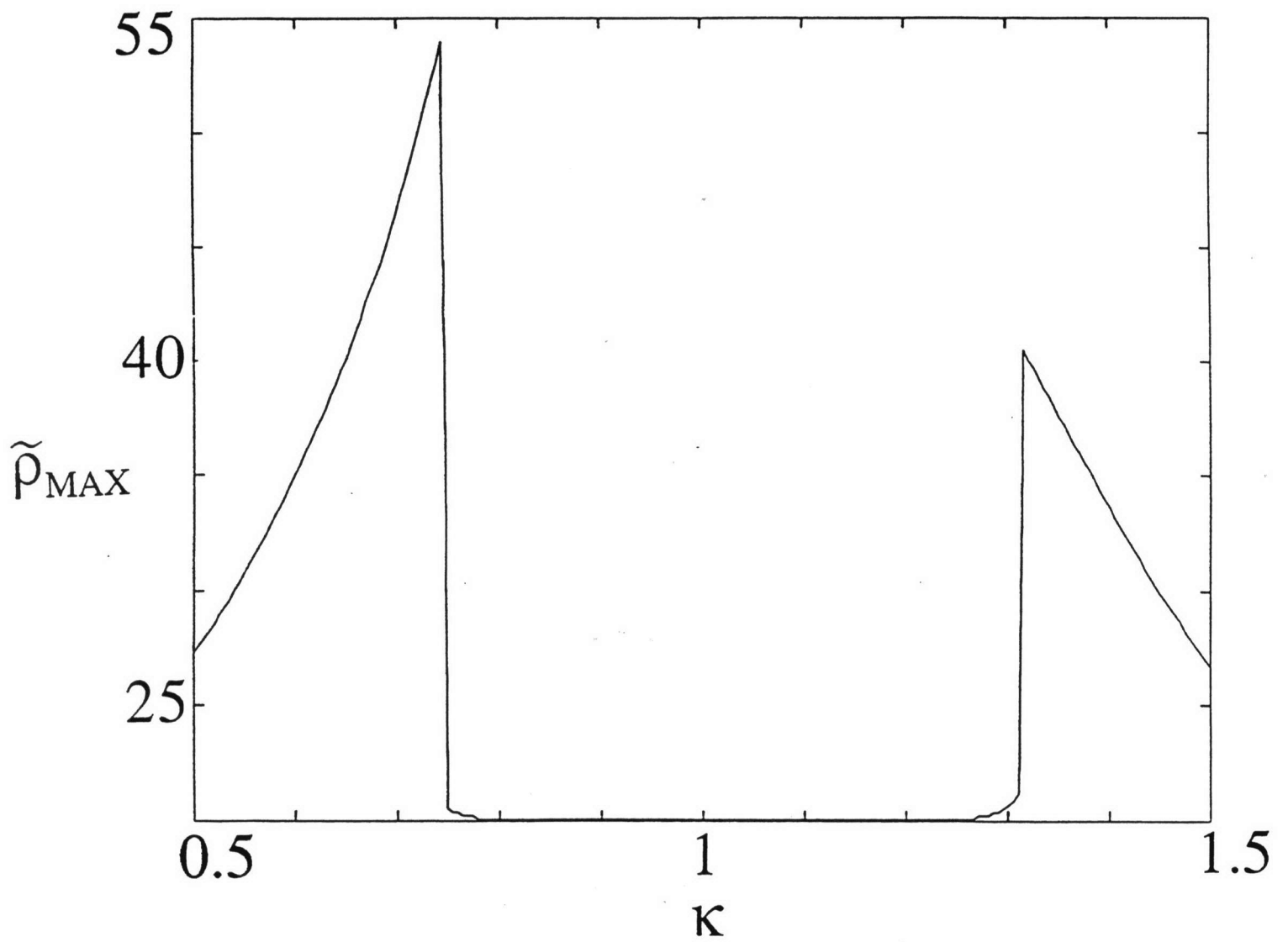


FIGURE 4 Benisti et al., "Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent Acceleration." Physics of Plasmas

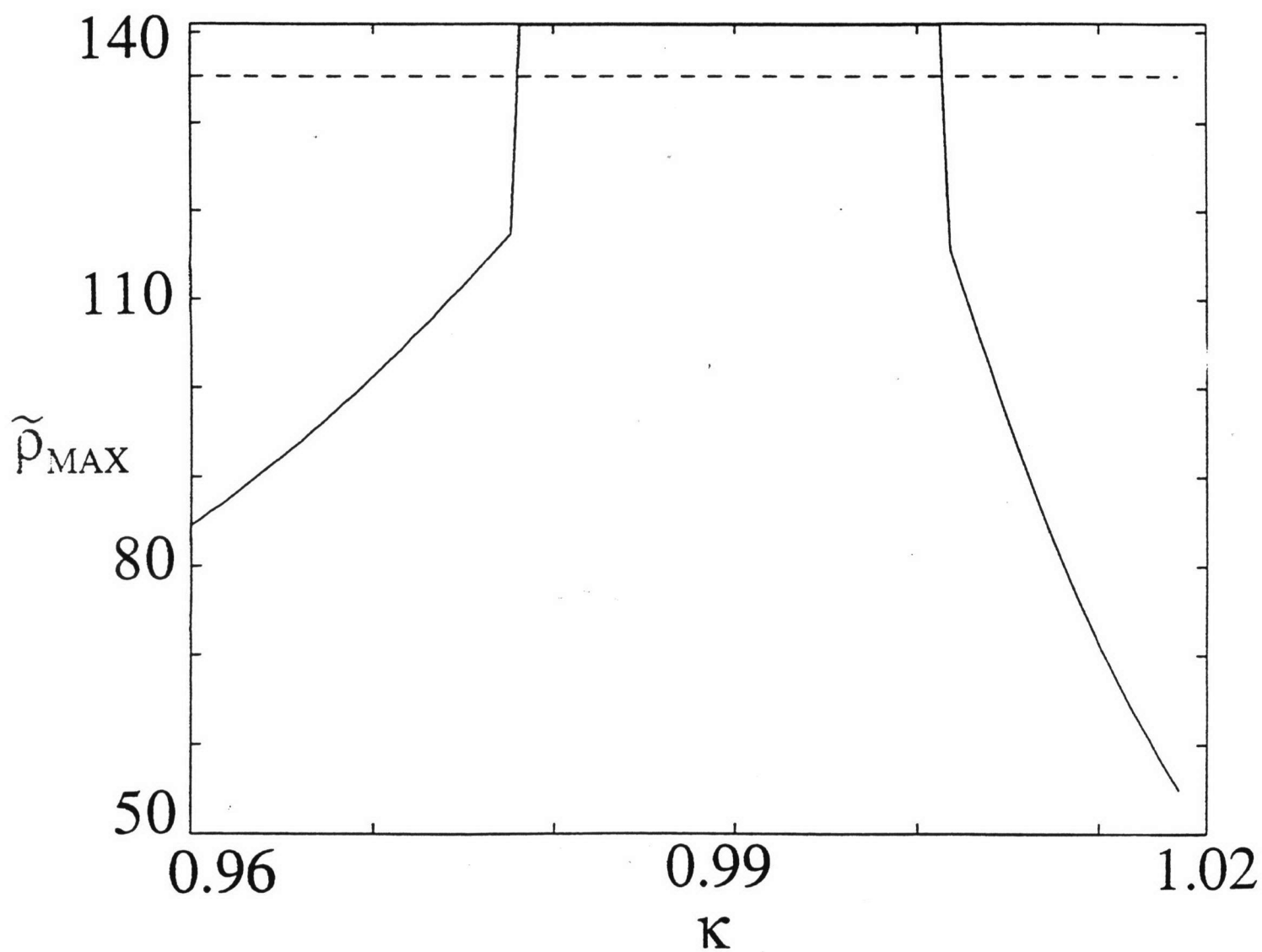


FIGURE 5 Benisti et al., "Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent Acceleration." Physics of Plasmas

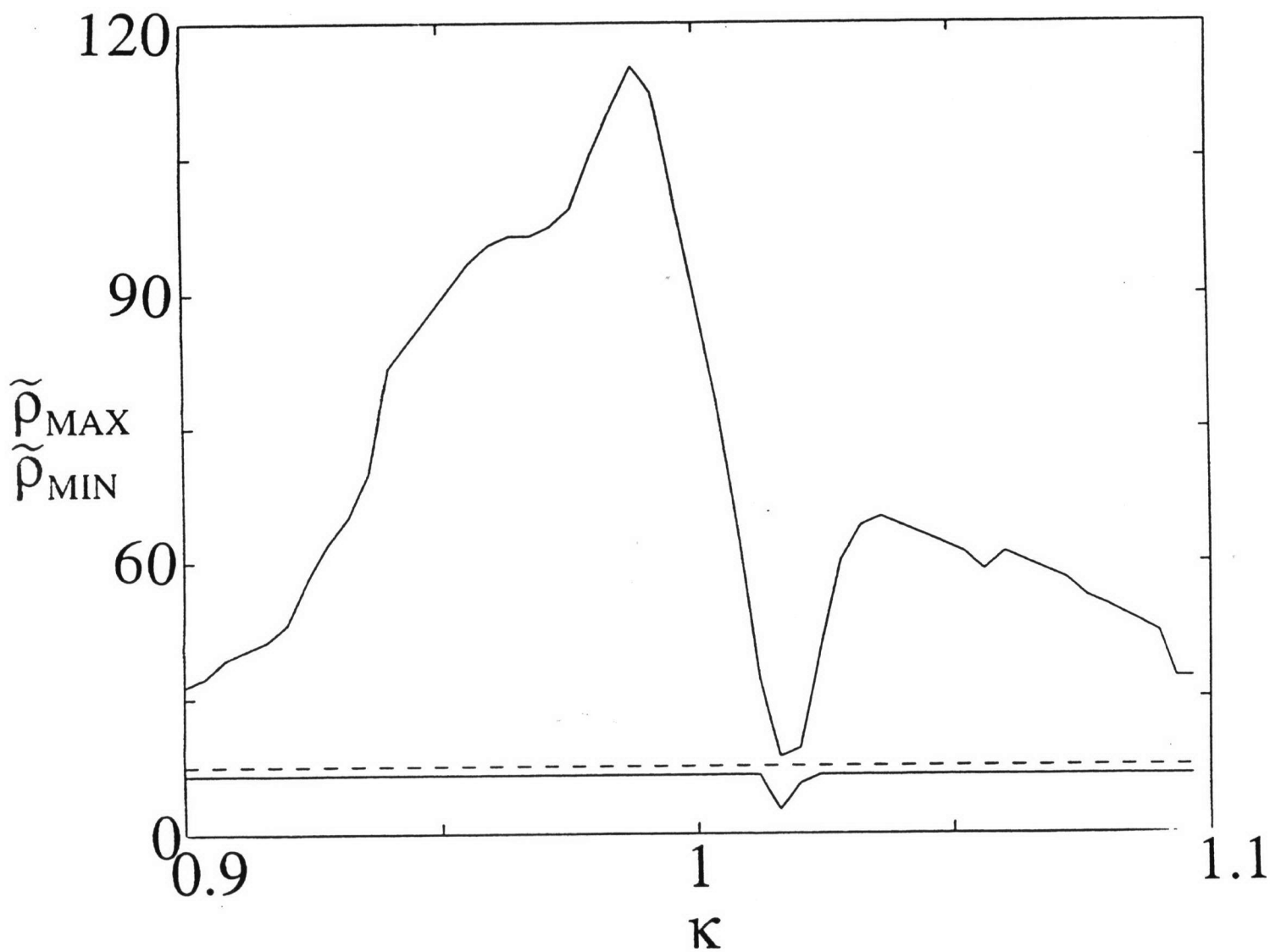


FIGURE 6 Benisti et al., "Ion Dynamics in Multiple Electrostatic Waves in a Magnetized Plasma - Part I: Coherent Acceleration." Physics of Plasmas